



## The Geometry of Contact Pseudo-Slant Submanifolds of a $(\kappa, \mu)$ –Contact Metric Manifold

Suleyman Dirik\*, Vezir Cetin

Amasya University, Faculty of arts and Science, Department of Mathematics, Amasya, Turkey

Accepted 20 November 2022

### Abstract

In this paper, the geometry of contact pseudo-slant submanifolds of a  $(\kappa, \mu)$  –contact metric manifold have been studied. The necessary and sufficient conditions for a submanifolds to be a contact pseudo- slant submanifolds of a  $(\kappa, \mu)$  –contact metric manifold are given.

**Keywords:**  $(\kappa, \mu)$  – contact metric manifolds, contact pseudo-slant submanifolds, totally umbilical, mixed geodesic.

### 1. Introduction

Because slant submanifolds are known to generalize invariant and anti-invariant submanifolds, many geometrs have expressed an interest in this research. Chen [8], [9] started this research on complex manifolds. Lotta[17] introduced slant immersions in an almost contact metric manifold. Carriezo defined a new class of submanifolds known as hemi-slant submanifolds (Also known as anti-slant or pseudo-slant submanifolds) [6] . The contact version of a pseudo-slant submanifold in a Sasakian manifold was then defined and studied by V. A. Khan and M. A. Khan. [13]. Later many geometers like ([10], [11], [12], [14], [16]) studied pseudo-slant submanifolds on various manifolds. Recently, M. Atçeken and S. Dirik studied contact pseudo-slant submanifold on various manifolds ([1],[11],[12]). Recently studied; Venkatesha, Srikantha N and siddesha M. S. On pseudo-slant submanifolds of  $(\kappa, \mu)$ –contact space forms and found some results [22].

In the light of the above studies, our article, the following is how this paper is structured: Section 2 includes some fundamental formulas and definitions of the  $(\kappa, \mu)$ –contact metric manifold and it is submanifolds. Section 3 we review some definitions and proves some basic results on the contact pseudo-slant submanifolds of the  $(\kappa, \mu)$ –contact metric manifold. The final section looks at the totally umbilical contact pseudo-slant in  $(\kappa, \mu)$ -contact metric manifolds.

A smooth  $(2n+1)$ -dimensional manifold  $\tilde{M}$  is said to be contact manifold if it has a global 1-form  $\eta$  satisfies  $\eta \wedge (d\eta)^m = 0$  everywhere on  $\tilde{M}$ . And a  $(2n+1)$ –dimensional almost contact manifold with almost contact structure  $(\varphi, \xi, \eta)$  made up of a  $(1,1)$  tensör field, a global 1–form and a characteristic vector field  $\xi$  satisfies [2], [3].

$$\varphi^2 X = -X + \eta(X)\xi \quad (1)$$

$$\eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta(\varphi X) = 0 \quad (2)$$

for all vector field X on  $\tilde{M}$ .

Let g be the compatible Riemannian metric with almost contact structure  $(\varphi, \xi, \eta)$  such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \quad (3)$$

$$g(\varphi X, Y) = -g(X, \varphi Y), \quad g(X, \xi) = \eta(X) \quad (4)$$

for all vector field X, Y on  $\tilde{M}$ . Then  $\tilde{M}$  equipped with an almost contact metric structure  $(\varphi, \xi, \eta, g)$  is referred to as an almost contact metric manifold.

Let  $\psi$  be the fundamental 2-form on  $\tilde{M}$  defined by  $\psi(X, Y) = g(X, \varphi Y)$  if  $\psi = d\eta$  almost contact metric structure transform into a contact metric structure transform into a contact metric structure.

In a contact metric manifold  $\tilde{M}$ , we known that the symmetric tensör h, defined by  $2h = \mathcal{L}_\xi \varphi$ , satisfied the following [2 ], where  $\mathcal{L}$  denotes the Lie

\* Corresponding author:  
Email: [slymndirik@gmail.com](mailto:slymndirik@gmail.com)

differentiation.

$$h\phi + \phi h = 0, h\xi = 0, \text{tr}(h) = \text{tr}(\phi h) = 0 \quad (5)$$

a contact metric manifold is a type of contact metric manifold. If the structural vector field  $\xi$  belongs to the  $(\kappa, \mu)$  nullity distribution defined by [3],  $M$  is said to be a  $(\kappa, \mu)$  contact metric manifold.

$$\begin{aligned} \tilde{M}(\kappa, \mu): p \rightarrow \tilde{M}_p(\kappa, \mu) &= \{W \in T_p \tilde{M} : \tilde{R}(X, Y)W \\ &= \kappa[g(Y, W)X - g(X, W)Y] \\ &+ \mu[g(Y, W)hX - g(X, W)hY]\} \end{aligned}$$

for all  $X, Y$  on  $\tilde{M}$ . Where  $\kappa, \mu$  are constants. We know that  $\tilde{M}$  exists in a  $(\kappa, \mu)$  -contact metric manifold.

$h^2 = (\kappa - 1)\phi^2$  and therefore  $\kappa \leq 1$ . If  $\kappa=1$  then  $\tilde{M}$  becomes a Sasakian manifold. Moreover for a  $(\kappa, \mu)$ -contact metric manifold  $\tilde{M}$  of dimensional  $(2m+1)$  and for all  $X, Y \in \Gamma(T\tilde{M})$ , we have [2].

$$(\tilde{\nabla}_X \phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX) \quad (6)$$

where  $\tilde{\nabla}$  denotes the Riemannian connection with respect to  $g$ . From (6), taking instead of  $\xi$

$$\tilde{\nabla}_X \xi = -\phi X - \phi hX \quad (7)$$

Let  $M$  be a submanifold of a contact metric manifold  $\tilde{M}$  with the same symbol  $g$  for the induced metric. Then the Gauss and Weingarten formulae are given by;

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y) \quad (8)$$

$$\tilde{\nabla}_X V = -A_V X + \nabla_X^\perp V \quad (9)$$

where  $\nabla$  and  $\nabla^\perp$  are induced connections on the tangent bundle  $TM$  and  $T^\perp M$  of  $M$  respectively,  $\sigma$  and  $A_V$  are the second fundamental form and the shape operator with respect to  $V$  respectively. Then the shape operator  $A_V$  and the second fundamental form  $\sigma$  are inter-connected by;

$$g(A_V X, Y) = g(\sigma(X, Y), V) \quad (10)$$

for all  $X, Y \in \Gamma(TM)$  and  $V \in (T^\perp M)$ .

The mean curvature vector  $H$  of  $M$  is given by;

$$H = \frac{\text{tr}(\sigma)}{r} = \frac{1}{r} \sum_{i=1}^r \sigma(e_i, e_i) \quad (11)$$

where  $r$  is the dimension of  $M$  and  $\{e_1, e_2, \dots, e_r\}$  is the local orthonormal frame of  $M$ .

- If a submanifold is totally umbilical, it is said to be totally umbilical.

$$\sigma(X, Y) = g(X, Y)H \quad (12)$$

- If  $\sigma(X, Y) = 0$ , a submanifold is said to be totally geodesic, where for all  $X, Y \in \Gamma(TM)$ .
- If  $H = 0$ , a submanifold is said to be minimal.

### Contact Pseudo-Slant Submanifolds of a $(\kappa, \mu)$ -Contact Metric Manifold

Let  $M$  be a submanifold of a  $(\kappa, \mu)$  -contact metric manifold  $\tilde{M}$ . Then for any  $X \in \Gamma(TM)$ , we can write;

$$\phi X = EX + FX \quad (13)$$

where  $EX$  is the tangent component and  $FX$  is the normal component of  $\phi X$ .

Also, for any  $V \in (T^\perp M)$ ,  $\phi V$  can be written in the following way;

$$\phi V = BV + CV, \quad (14)$$

where  $BV$  and  $CV$  are also the tangent and normal component of  $\phi V$ , respectively.

We can deduce from (13) and (14) that the tensor field,  $E, F, B$  and  $C$  are also anti-symmetric because  $\phi$  is anti-symmetric. By using (1), (13) and (14), we obtain;

$$E^2 + BF = -I + \eta\phi\xi, \quad FE + CF = 0 \quad (15)$$

$$C^2 = -I - FB, \quad EB + BC = 0 \quad (16)$$

Furthermore, we can deduce from (4), (13) and (14) that  $E$  and  $C$  are skew symmetric tensor fields.

$$g(EX, Y) = -g(X, EY) \quad (17)$$

$$g(CX, Y) = -g(X, CY) \quad (18)$$

Also, we can obtain relation between  $F$  and  $B$  as;

$$g(FX, V) = -g(X, BV) \quad (19)$$

for all  $X, Y \in \Gamma(TM)$  and  $V \in (T^\perp M)$ .

Also defined are the covariant derivatives of the tensor fields  $E, F, B$ , and  $C$ .

$$(\nabla_X E)Y = \nabla_X EY - E\nabla_X Y, \quad (20)$$

$$(\nabla_X F)Y = \nabla_X^\perp FY - F\nabla_X Y \quad (21)$$

$$(\nabla_X B)V = \nabla_X BV - B\nabla_X^\perp V \quad (22)$$

and

$$(\nabla_X C)V = \nabla_X^\perp CV - C\nabla_X^\perp V \quad (23)$$

The covariant derivative of  $\phi$ ,  $\tilde{\nabla}\phi$  can be defined by

$$(\tilde{\nabla}_X \phi)Y = \tilde{\nabla}_X \phi Y - \phi \tilde{\nabla}_X Y$$

for all  $X, Y \in \Gamma(TM)$  and  $V \in (T^\perp M)$ . Where  $\tilde{\nabla}$  is the Riemannian connection on  $\tilde{M}$ .

**Lemma:** Let  $M$  be a submanifold of a  $(\kappa, \mu)$  – contact metric manifold  $\tilde{M}$ . Then;

$$\begin{aligned}
 (\nabla_X E)Y &= A_{FY}X + B\sigma(X, Y) + g(hX, Y)\xi - \eta(Y)(X + hX) & (24) \\
 (\nabla_X F)Y &= C\sigma(X, Y) - \sigma(X, EY) & (25) \\
 (\nabla_X B)V &= A_{CV}X - EA_VX & (26) \\
 (\nabla_X C)V &= -\sigma(X, BV) - FA_VX & (27)
 \end{aligned}$$

for all  $X, Y \in \Gamma(TM)$  and  $V \in (T^\perp M)$ . Using (5), (6), (7), and (8), we have that  $\xi$  is tangent to  $M$ .

$$\begin{aligned}
 \nabla_X \xi &= -EX - EhX, & (28) \\
 \sigma(X, \xi) &= -FX - FhX & (29)
 \end{aligned}$$

for all  $X \in \Gamma(TM)$ . Let us some definitions of classes submanifolds.

- If  $F$  is identically zero in (13), then the submanifold is invariant, i.e.,  $\varphi X \in \Gamma(TM)$ ,  $\forall X \in \Gamma(TM)$ .
- If  $E$  is identically zero in (13), then the submanifold is anti-invariant, i.e.,  $\varphi X \in \Gamma(T^\perp M)$ ,  $\forall X \in \Gamma(TM)$ .
- If there is a constant angle  $\theta \in [0, \frac{\pi}{2}]$  between  $\varphi X$  and  $TM$  for all nonzero vector  $X$  tangent to  $M$  at  $X$ , the manifold is called slant.
- If there are distribution  $D_\theta, D^\perp$ , is there a contact pseudo-slant submanifold. Such that
  - 1) orthogonal direct composition is allowed in
 
$$TM = D^\perp \oplus D_\theta, \zeta \in D_\theta$$
  - 2)  $D_\theta$  is slant distribution with slant angle ,  $\theta = \frac{\pi}{2}$
  - 3)  $D^\perp$  is an anti-invariant distribution [ 13].

We can see from the definitions that a slant submanifold is a generalization of invariant (if  $\theta = 0$ ) and anti-invariant (if  $\theta = \frac{\pi}{2}$ ) submanifolds. A proper slant submanifold is submanifold that is neither invariant non anti invariant i. e.,  $(\theta = \frac{\pi}{2})$ . As a result, the following theorem characterized slant submanifolds of almost contact metric manifold in general;

**Theorem 1:** [5] Let  $M$  be a slant submanifolds of an almost contact metric manifold  $\tilde{M}$  such that  $\xi \in \Gamma(TM)$ , then,  $M$  is slant submanifold if and only if there exist a constant  $\lambda \in [0, 1]$  such that

$$E^2 = -\lambda(1 - \eta \otimes \xi) \tag{30}$$

furthermore, in this case, if  $\theta$  is the slant angle of  $M$ . Then it satisfies  $\lambda = \cos^2 \theta$ .

**Corollary1 :** [5] Let  $M$  be a slant submanifolds of an almost contact metric manifold  $\tilde{M}$ . Then for all  $X, Y \in \Gamma(TM)$  we have;

$$\begin{aligned}
 g(EX, EY) &= \cos^2 \theta \{g(X, Y) - \eta(X)\eta(Y)\} & (31) \\
 g(FX, FY) &= \sin^2 \theta \{g(X, Y) - \eta(X)\eta(Y)\}. & (32)
 \end{aligned}$$

If we denote the orthogonal complementary of  $\varphi TM$  in  $T^\perp M$  by  $\nu$ , then the normal bundle  $T^\perp M$  can be decomposed as follows;

$$T^\perp M = FD_\theta \oplus FD^\perp \oplus \nu, \quad FD_\theta \perp FD^\perp.$$

From the above decomposition one can see that  $g(X, Y) = 0$ , for all  $X \in \Gamma(D_\theta)$  and  $Y \in \Gamma(D^\perp)$ . And thus  $g(\varphi X, \varphi Y) = g(\varphi X, FY) = g(FX, FY) = g(X, Y) = 0$ .

We also known that  $B\sigma(X, Y) = 0$ , thus (24) reduces to

$$(\nabla_X E)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX) \tag{33}$$

for all  $X, Y \in D_\theta$ . As a result, if the ambient manifold  $M$  is  $(\kappa, \mu)$ -contact metric manifold, the induced structure  $E$  is a  $(\kappa, \mu)$ -contact metric structure on  $M$ .

**Definition 1 :** A contact pseudo-slant submanifold  $M$  of a  $(\kappa, \mu)$ -contact metric manifold  $\tilde{M}$  is said to be mixed-geodesic submanifold if  $\sigma(X, Y) = 0$  for all  $X \in \Gamma(D_\theta)$  and  $Y \in \Gamma(D^\perp)$ .

**Theorem 2.** Let  $M$  be proper contact pseudo-slant submanifold of a  $(\kappa, \mu)$ -contact metric manifold  $\tilde{M}$ . If  $B$  is parallel, then  $M$  is either mixed geodesic or anti-invariant submanifold.

**Proof:** For all  $X \in \Gamma(D_\theta)$  and  $Y \in \Gamma(D^\perp)$ , from (25) and (26)  $B$  parallel if and only if  $F$  parallel, so  $\nabla F = 0$ . This implies

$$C\sigma(X, Y) - \sigma(X, EY) = 0$$

Replacing  $X$  by  $EX$  in the above equation, we have;

$$C\sigma(EX, Y) - \sigma(EX, EY) = 0$$

for  $Y \in \Gamma(D^\perp)$ ,  $EY=0$ . Hence

$$C\sigma(EX, Y) = 0$$

Replacing X by EX in the above equation we get

$$C \sigma(E^2X, Y) = -C \cos^2\theta\sigma(X, Y) = 0$$

Hence we have either  $\sigma(X, Y) = 0$  (M is mixed geodesic) or  $\theta = \frac{\pi}{2}$  (M is anti-invariant).

**Theorem 3.** Let M be a contact pseudo-slant submanifold of a  $(\kappa, \mu)$ -contact metric manifold  $\tilde{M}$ . If F is parallel, on  $D_\theta$ , then either M is a  $D_\theta$ -geodesic submanifold or  $\sigma(X, Y)$  is an eigen vector of  $E^2$  with eigen values  $-\cos^2\theta$ , for all  $X, Y \in \Gamma(D_\theta)$ .

**Proof:** For all  $X, Y \in \Gamma(D_\theta)$ , from( 25 ), we have

$$C \sigma(X, Y) - \sigma(X, EY) = 0 \tag{34}$$

on the other hand, since  $D_\theta$  is slant, we have;

$$C \sigma(X, Y - \eta(Y)\xi) - \sigma(X, EY) = 0$$

that is

$$C \sigma(X, Y - \eta(Y)\xi) = \sigma(X, EY) \tag{35}$$

Now applying C to (35), we obtain

$$C^2 \sigma(X, Y - \eta(Y)\xi) = C\sigma(X, EY) \tag{36}$$

by interchanging of Y and EY in (34), we get

$$\sigma(X, E^2Y) = C\sigma(X, EY) \tag{37}$$

Hence, using (30), (36) and (37) we obtain;

$$C^2 \sigma(X, Y - \eta(Y)\xi) = C\sigma(X, EY) = \sigma(X, E^2Y) = -\cos^2\theta \sigma(X, Y - \eta(Y)\xi).$$

This implies that  $\sigma$  vanishes on  $D_\theta$  or  $\sigma$  is eigen vector of  $C^2$  with eigenvalues  $= -\cos^2\theta$ .

**Theorem 4.** Let M be a contact pseudo-slant submanifold of a  $(\kappa, \mu)$ -contact metric manifold  $\tilde{M}$ . Then anti-invariant distribution  $D^\perp$  is always integrable.

**Proof:** For all  $Z, W \in \Gamma(D^\perp)$ , from (6), we have

$$\begin{aligned} (\tilde{\nabla}_Z\phi)W &= g(Z + hZ, W)\xi - \eta(W)(Z + hZ) \\ &= g(Z + hZ, W)\xi. \end{aligned}$$

By using (8), (9) , (13) and (14), we have

$$\tilde{\nabla}_Z\phi W - \phi\tilde{\nabla}_Z W - g(Z + hZ, W)\xi = 0$$

$$\tilde{\nabla}_ZFW - \phi(\nabla_ZW + \sigma(Z, W)) - g(Z + hZ, W)\xi = 0$$

$$\begin{aligned} -A_{FW}Z + \nabla_Z^\perp W - E\nabla_ZW - F\nabla_ZW - B\sigma(Z, W) \\ - C\sigma(Z, W) - g(Z + hZ, W)\xi = 0 \end{aligned}$$

Comparing the tangent components, we have;

$$-A_{FW}Z - E\nabla_ZW - B\sigma(Z, W)g(Z + hZ, W)\xi = 0 \tag{38}$$

interchanging Z and W, we get

$$-A_{FZ}W - E\nabla_WZ - B\sigma(W, Z) - g(W + hW, Z)\xi = 0 \tag{39}$$

Subtracting equation (39) from (38) and using the fact that  $\sigma$  and h are symmetric , we get;

$$\begin{aligned} A_{FW}Z - A_{FZ}W + E[\nabla_ZW - \nabla_WZ] &= 0 \\ A_{FW}Z - A_{FZ}W + E[Z, W] &= 0 \end{aligned}$$

$$E[W, Z] = A_{FW}Z - A_{FZ}W \tag{40}$$

On the other hand, for all  $U \in \Gamma(TM)$ . By using (6), (8) (9) and (10), we have;

$$\begin{aligned} g(A_{FW}Z - A_{FZ}W, U) &= g(\sigma(Z, U), FW) - \\ g(\sigma(W, U), FZ) &= g(\sigma(Z, U), FW) - g(\tilde{\nabla}_UW, FZ) \\ &= g(\sigma(Z, U), FW) + g(\phi\tilde{\nabla}_UW, Z) \\ &= g(\sigma(Z, U), FW) + g(\tilde{\nabla}_U\phi W - (\tilde{\nabla}_U\phi)^W, Z) \\ &= g(\sigma(Z, U), FW) + g(-A_{FW}U + \nabla_U^\perp FW, Z) \\ &\quad - g(g(U + Hu, W)\xi, Z) \\ &= g(\sigma(Z, U), FW) - g(A_{FW}U, Z) \\ &= g(\sigma(Z, U), FW) \\ &= g(\sigma(Z, U), FW) = 0 \end{aligned}$$

Here

$$A_{FW}Z = A_{FZ}W.$$

From (40),  $E[W, Z] = 0$  notice that  $D^\perp$  is integrable if  $[W, Z] \in \Gamma(D^\perp)$ .  $D^\perp$  is anti-invariant. i. e.  $\phi D^\perp \subseteq T^\perp M$  so  $EZ=0, EW=0$  and  $E[W, Z] = 0$ .

**Theorem 5.** Let M be a contact pseudo-slant submanifold of a  $(\kappa, \mu)$ -contact metric manifold  $\tilde{M}$ . Then the slant distribution  $D_\theta$  is integrable if and only if

$$P_1\{\nabla_X EY - A_{FY}X - E\nabla_YX - B\sigma(X, Y) + \eta(Y)(X + hX)\} = 0$$

for all  $X, Y \in \Gamma(D_\theta)$ .

**Proof:** We denote by  $P_1$  the projection on  $D^\perp$ . For all  $X, Y \in \Gamma(D_\theta)$  from (6), we have;

$$(\tilde{\nabla}_X\phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX)$$

On applying (8), (9), (13) and (14), we get

$$\tilde{\nabla}_X \phi Y - \phi \tilde{\nabla}_X Y - g(X + hX, Y)\xi + \eta(Y)(X + hX) = 0$$

$$\tilde{\nabla}_X EY + \tilde{\nabla}_X FY - \phi(\nabla_X Y + \sigma(X, Y)) - g(X + hX, Y)\xi + \eta(Y)(X + hX) = 0$$

$$\nabla_X EY + \sigma(X, EY) - A_{FY}X + \nabla_X^\perp FY - E\nabla_X Y - F\nabla_X Y - B\sigma(X, Y) - C\sigma(X, Y) - g(X + hX, Y)\xi + \eta(Y)(X + hX) = 0$$

Comparig the tangential components;

$$\nabla_X EY - A_{FY}X - E\nabla_X Y - B\sigma(X, Y) - g(X + hX, Y)\xi + \eta(Y)(X + hX) = 0$$

$$\nabla_X EY - A_{FY}X - E\nabla_Y X + E\nabla_Y X - E\nabla_X Y - B\sigma(X, Y) - g(X + hX, Y)\xi + \eta(Y)(X + hX) = 0$$

$$E[X, Y] = \nabla_X EY - A_{FY}X - E\nabla_Y X - B\sigma(X, Y) - g(X + hX, Y)\xi + \eta(Y)(X + hX) \tag{41}$$

Hence applying  $P_1$  to both sider of (41) equation, we conclude our theorem for all  $X, Y \in \Gamma(D_\theta)$ ,  $[X, Y] \in \Gamma(D_\theta)$ , so  $P_1 E[X, Y]=0$ .

**Theorem 6.** Let  $M$  be a totally umbilical contact pseudo-slant submanifold of a  $(\kappa, \mu)$  -contact metric manifold  $\tilde{M}$ . Then at least one of the following satements is true.

- 1-  $M$  is proper contact pseudo-slant submanifold.
- 2-  $H \in \Gamma(\nu)$ .
- 3-  $\text{Dim}(D^\perp) = 1$ .

**Proof:** Let  $X \in \Gamma(D^\perp)$  and using (6), we obtain

**References**

[1] M. Atçeken and S. Dirik , On the geometry of pseudo-slant submanifold of a Kenmotsu manifold, Gulf Journal of Mathematics, 2(2) (2014), 51-66.  
 [2] D. E. Blair, Contact Manifolds in Riemannian Geometry: Lecture Notes in Mathematics, 509, Springer, Berlin (1976).  
 [3] D. E. Blair, Riemannian Geometry of Contact and Symplectic Manifolds, Birkhäuser, Boston (2002).  
 [4] D. E. Blair, T. Koufogiorgos, B. J. Papantoniou, Contact metric manifolds satisfying a nullity condition, Israel Journal of Mathematics 91, (1995), 189-214.  
 [5] J. L. Cabrerizo, A. Carriazo, L. M. Fernandez, M. Fernandez, Slant submanifolds in Sasakian manifolds, Glasgow Mathematical Journal, 42, (2000), 125-138.

$$(\tilde{\nabla}_X \phi)X = g(X + hX, X)\xi - \eta(X)(X + hX)$$

On applying (8), (9), (13) and (14), we get;

$$\tilde{\nabla}_X FX - \phi(\nabla_X X + \sigma(X, X)) - g(X + hX, X)\xi = 0$$

$$-A_{FX}X + \nabla_X^\perp FX - F\nabla_X X - B\sigma(X, X) - C\sigma(X, X) - g(X + hX, X)\xi = 0$$

Comparig the tangential components

$$-A_{FX}X + B\sigma(X, X) + g(X + hX, X)\xi = 0$$

Taking the product by  $W \in \Gamma(D^\perp)$ , we obtain

$$g(A_{FX}X, W) + g(B\sigma(X, X), W) = 0$$

Since  $M$  is totally umbilical submanifold, we obtain

$$g(A_{FX}W, X) + g(B\sigma(X, X), W) = 0$$

$$g(\sigma(W, X), FX) - g(\sigma(X, X), FW) = 0$$

$$g(W, X)g(H, FX) - g(X, X)g(H, FW) = 0$$

$$g(X, X)g(BH, W) - g(W, X)g(BH, X) = 0$$

that is

$$g(BH, W)X - g(BH, X)W = 0$$

Here  $BH$  is either zero or  $X$  and  $W$  are linearly dependent vector fields. If  $BH \neq 0$ , than  $\text{dim}(D^\perp) = 1$ . Otherwise  $H \in \Gamma(\mu)$ . Since  $D_\theta \neq 0$   $M$  is contact pseudo-slant submanifold. Since  $\theta \neq 0$  and  $d_1 d_2 \neq 0$  proper contact pseudo-slant submanifold.

[6] A. Carriazo, New Devolpments in Slant Submanifolds Theory, Narosa publishing House, New Delhi, India, (2002).  
 [7] A. Carriazo, V. Martin-Molina and M. M. Tripathi, Generalized  $(\kappa, \mu)$ -space forms, Mediterranean Journal of Mathematics, 10, (2013), 475-496, doi: 10.1007/s00009-012-0196-2.  
 [8] B. Y. Chen, Geometry of slant submanifolds, Katholieke Universiteit Leuven, Leuven, (1990).  
 [9] B.Y. Chen, Slant immersions, Bulletin of the Australian Mathematical Society, 41, (1990), 135-147.  
 [10] U.C. De and A. Sarkar, On Pseudo-slant submanifolds of trans-Sasakian manifolds, Proceedings of the Estonian Academy of Sciences, 60, 1(2011), 1-11, doi: 10.3176/proc.2011.1.01.  
 [11] S. Dirik, M. Atçeken, Pseudo-slant

- submanifolds in Cosymplectic space forms, *Acta Universitatis Sapientiae: Mathematica*, 8, 1(2016), 53-74, doi: 10.1515/ausm-2016-0004.
- [12] S. Dirik, M. Atçeken, U. Yildirim, Pseudo-slant submanifold in Kenmotsu space forms, *Journal of Advances in Mathematics*, 11, 10(2016), 5680-5696.
- [13] V. A. Khan and M. A. Khan, Pseudo-slant submanifolds of a Sasakian manifold, *Indian Journal of pure and applied Mathematics*, 38, 1(2007), 31-42.
- [14] M. A. Khan, Totally umbilical Hemi-slant submanifolds of Cosymplectic manifolds, *Mathematica Aeterna*, 3, 8(2013), 645-653.
- [15] T. Koufogiorgos, Contact Riemannian manifolds with constant  $\varphi$ -sectional curvature, *Tokyo Journal of Mathematics*, 20, 1(1997), 13-22.
- [16] B. Laha and A. Bhattacharyya, Totally umbilical Hemislant submanifolds of LP-Sasakian Manifold, *Lobachevskii Journal of Mathematics*, 36, 2(2015), 127-131, doi: 10.1134/S1995080215020122.
- [17] A. Lotta, Slant submanifolds in contact geometry, *Bulletin of Mathematical Society Romania*, 39, (1996), 183-198.
- [18] S. Phan, On geometry of warped product pseudo-slant submanifolds on generalized Sasakian space forms, *Gulf Journal of Mathematics* 9(1), (2020), 42-61.
- [19] M.S. Siddesha and C. S. Bagewadi, On Slant submanifolds of  $(\kappa, \mu)$ -contact manifold, *Differential Geometry-Dynamical Systems*, 18, (2016), 123-131.
- [20] S. Tanno, Ricci curvatures of contact Riemannian manifolds, *Tohoku Mathematical Journal*, 40, (1988), 441-448.
- [21] P. Uygun and M. Atçeken, On a  $(\kappa, \mu)$ -paracontact manifold satisfying some conditions on the projective curvature tensor, *Gulf Journal of Mathematics*, 11(2) (2021), 27-35.
- [22] N. Venkatesha, Srikantha and M.S.Siddesha, On pseudo-slant submanifolds of  $(\kappa, \mu)$ -contact space forms, *Palestine Journal of Mathematics*, 8(2)(2019), 248-257.
- [23] K. Yano and S. Ishihara, Invariant submanifolds of almost contact manifolds, *Kodai Math. Sem. Rep.*, 21, (1969), 350-364.
- [24] R. Sarı, "Some properties curvature of Lorentzian Kenmotsu manifolds." *Applied Mathematics and Nonlinear Sciences* 5.1 (2020): 283-292.
- [25] R. Sarı, and Aysel Turgut Vanlı. Slant submanifolds of a Lorentz Kenmotsu manifold. *Mediterranean Journal of Mathematics* 16.5 (2019): 1-17.
- [26] P. Uygun, S. Dirik M. Atçeken, and T. Mert. The geometry of invariant submanifold of a  $(\kappa, \mu)$ -paracontact metric manifold, *International Journal of Poineering Tecnology end Engineering*. 1(2), 42-45.