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Submersions of Generic Submanifolds of a Para Kenmotsu Manifold

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Abstract

In this paper, we investigate the submersions of generic submanifolds of a para Kenmotsu manifold onto almost contact manifold. We also obtain the decomposition theorems for such submersions and derive the relation between curvatures.

Keywords: Submersions, generic submanifolds, para Kenmotsu manifold

1. Introduction

Para complex geometry is the geometry which is related to the algebra of para complex numbers [2]. The para complex structures is defined on a smooth manifold M with the endomorphism $J^2 = I$ such that the 1-eigen distributions are integrable and has the same dimension. There are some differences between para complex geometry and complex geometry. The product of two manifolds of the same dimension is the natural example of para complex manifold . These manifolds also have some applications in physics, they are related to supersymmetric field theories with Euclidean space time.

In 1985Willams and Kaneyuki studied the almost para-contact structure on (2n+1)-dimensional pseudo-Riemannian manifold M and showed the almost paracomplex structure on the product manifold ofM and real line R [3]. Also Zamkovoy [19] studied an almost paracontact metric manifold and their sub classes. Similar to contact manifolds some classes of para-contact manifolds are defined and this notion studied by geometers. One of them is para-Kenmotsu manifolds. Some authors were introduced para Kenmotsu manifold and generic submanifolds [1,4,12,15,16,17,18].

One way to compare two manifolds is to define smooth maps from one manifold to another. One of these maps is submersion, the rank of the map is equal to the dimension of the target manifold. An isometric submersion is called a Riemannian submersion. Riemannian submersion between Riemannian submanifolds was first introduced by O'Neill [8].

In 1981, Kobayashi [6] investigated submersion of CR-submanifold of a Kaehler manifold onto almost Hermitian manifold. After, Deshmekh et al. studied properties curvature of this submersions [4]. In 1989, Papaghuic [9] studied the submersion of generic submanifolds of a Sasakian manifold. Submersion of generic submanifolds of trans-Sasakian manifold were studied by Shaid et al [11]. Moreover, there are many papers about these subject in literatüre [5,13,14].

In this paper, we study submersions of generic submanifold of Kenmotsu manifold onto almost contact manifold. We have been shown that in submersion of a generic submanifold of a Kenmotsu manifold onto an almost contact metric manifold, an almost contact metric manifold is a Kenmotsu manifold. Moreover, we investigated decomposition theorems and curvature relation.

2. Preliminaries

Let M be a *(2n+1)*-dimensional differentiable manifold endowed with a (φ, ξ, η, g), where φ is (1,1)-tensor field, ξ is a vector field, η is a 1-form, and g is a pseudo-Riemannian metric. If for all $X, Y \in \Gamma(TM)$ following conditions are satifeied then $\overline{\mathrm{M}}$ called almost para contact metric manifold:

$$\varphi^2 X = X - \eta(X)\xi, \quad \eta(\xi) = 1 \tag{1}$$
$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2}$$

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The contact metric manifold M is called a para Kenmotsu manifold if it satisfies the condition

$$(\overline{\nabla}_X\varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X \qquad (3)$$

for all $X, Y \in \Gamma(TM)$ where $\overline{\nabla}$ is Levi-Civita connection on M.

Let M be an n-dimensional isometrically immersed submanifold of para Kenmotsu manifold $\overline{M}$ and tangent to ξ and suppose $\overline{\nabla}$ (resp. ∇) be the covariant differentiation with respect to the Levi-Civita connection on $\overline{M}$ (resp. M). The Gauss and Weingarten formulae for M are respectively given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y) \qquad (4)$$

and

$$\overline{\nabla}_X N = -A_N X + \nabla_X^{\perp} N \qquad (5)$$

for $X, Y \in \Gamma(TM)$, $N \in \Gamma(T^{\perp}M)$, where h (resp. A) is the second fundamental form (resp. tensor) of M in $\overline{M}$ and $\nabla^{\perp}$ denotes the operator of the normal connection. Moreover we have

$$g(h(X,Y),N) = g(A_N X, Y) \qquad (6)$$

The curvature tensor R of the submanifold M is related to the curvature tensor $\bar{R}$ of $\bar{M}$ by the following Gauss formula

$$\bar{R}(X,Y,Z,W) = R(X,Y,Z,W) - g\big(h(Y,Z),h(X,W)\big) + g(h(X,Z),h(Y,W)) \qquad (7)$$

Definition 2.1. Let M be submanifold of para Kenmotsu manifold $\overline{M}$ with characteristic vector ξ is tangent to M. If the maximal invariant subspace under φ, orthogonal to ξ to $T_x(M)$.

$$H_x = T_x(M) \cap \varphi T_x(M), \quad x \in M$$

defines a differentiable distribution of $T_x(M)$, then M is called generic submanifold of $\overline{M}$.
For a generic submanifold M in a para Kenmotsu manifold $\overline{M}$, the orthogonal complementary distribution ${H_x}^{\perp}$ called the purely real distribution, if it satisfies

$$H_x \perp {H_x}^{\perp}, \ TH_x \subset {H_x}^{\perp}$$
$$H_x \cap \varphi {H_x}^{\perp} = 0.$$

Let μ_x be the vector space of holomorphic normal vectors to M at x, or simply the holomorphic normal space of M at x, i.e.,

$$\mu_x = {T_x}^{\perp} \cap \varphi {T_x}^{\perp}.$$

Then μ_x defines a differentiable vector subbundle of ${T_x}^{\perp}M$. It is easy to verify that

$$TM^{\perp} = T(H^{\perp}) \oplus \mu, \qquad tTM^{\perp} = H^{\perp}$$

and

$$g(T(H^{\perp}),\mu) = 0.$$

A vector subbundle of μ of the normal bundle $TM^{\perp}$ is said to be parallel (in the normal bundle) if

$$\nabla_x{}^{\perp}U \in \mu$$

for any $X \in \Gamma(TM)$ and any local cross-section U in μ.

The projection of TM to H and $H^{\perp}$ are denoted by P and F respectively i.e., for any $X \in TM$ we have

$$\varphi X = PX + FX \qquad (8)$$

The normal bundle to M has the decomposition

$$T^{\perp}M = \varphi H^{\perp} \oplus \mu.$$

For any $U \in \Gamma(T^{\perp}M)$, we put

$$U = pU + qU \qquad (9)$$

where $pU \in \Gamma(\varphi H^{\perp})$, $qU \in \Gamma(\mu)$. Making use of the above equation, we may write

$$\varphi U = \varphi pU + \varphi qU , \quad U \in \Gamma(T^{\perp}M) , \ \varphi pU \in \Gamma(H^{\perp}) , \quad \varphi qU \in \Gamma(\mu).$$

Now we determine Riemanniann submersion.
Let $\pi: (M^n, g_M) \to (B^b, g_B)$ be a submersion between two Riemannian manifolds. Then π said to be Riemannian submersion if
i) π has maximal rank
ii) The diferantial π_* preserves the lengths of horizontal vector.
On the other hand, a Riemannian submersion $\pi: M \to B$ determines tensor fields T and A on B such that,

$$T(E,F) = T_E F = h\nabla_{vE} vF + v\nabla_{vE} hF$$

$$A(E,F) = A_E F = v\nabla_{hE} hF + h\nabla_{hE} vF$$

for any $E, F \in \Gamma(TM)$.
Moreover, [11] the curvature tensors R of the

connection ∇, we have

$$\begin{aligned} R(X,Y,V,W) = {} & g\big((\nabla_V A)(X,Y),W\big) \\ & - g\big((\nabla_W A)(X,Y),V\big) \\ & + g(\mathcal{A}_X V,\mathcal{A}_Y W) - \\ g(\mathcal{A}_X W,\mathcal{A}_Y V) - {} & g(\mathcal{T}_V X,\mathcal{T}_W Y) + g(\mathcal{T}_W X,\mathcal{T}_V Y) \end{aligned} \tag{10}$$

On the other hand, $\pi^{-1}(k)$ is an $(n-b)$-dimensional submanifold of M, for each $k \in M$. The submanifolds $\pi^{-1}(k)$ are called fibers. Moreover, vector fields tangent to fibers are called vertical and vector fields orthogonal to fibers are horizantal. A vector field X on M is called basic if X is horizontal and $\pi_* X_p = X'_{\pi_{*(q)}}$ for all $q \in M$. We determine that V and H define projections $ker\pi_*$ and $(ker\pi_*)^{\perp}$, respectively.

Lemma 2.2. [10] Let X, Y be basic vector fields on M. Then

(i) $g(X,Y) = g'(X_*,Y_*) \circ \pi$,

(ii) The component $h([X,Y]) + \eta([X,Y])\xi$ of $[X,Y]$ is a basic vector field and corresponds to $[X_*,Y_*]$, i.e., $\pi_*(h([X,Y]) + \eta([X,Y])\xi = [X_*,Y_*]$,

(iii) $[U,X] \in H^{\perp}$ for any $U \in \Gamma(H^{\perp})$,

(iv) $h(\nabla_X Y) + \eta(\nabla_X Y)\xi$ is a basic vector field corresponding to $\nabla^*_X Y_*$, where ∇^* denotes the Levi-Civita connection on B.

For basic vector fields on M, we define the operator $\widetilde{\nabla}^*$ corresponding to ∇^* by setting $\widetilde{\nabla}^*_X Y = h(\nabla_X Y) + \eta(\nabla_X Y)\xi$ for $X, Y \in \Gamma(H \oplus \{\xi\})$. By (iv) of lemma 2.1., $\widetilde{\nabla}^*_X Y$ is a basic vector field and we have

$$\pi_*(\widetilde{\nabla}^*_X Y) = \nabla^*_{X_*} Y_* \ . \tag{11}$$

Define the tensor field C by

$$\nabla_X Y = \widetilde{\nabla}^*_X Y + C(X,Y), \quad X,Y \in \Gamma(H \oplus \{\xi\}) \tag{12}$$

where $C(X,Y)$ is the vertical part of $\nabla_X Y$. It is known that C is skew-symmetric and satisfies

$$C(X,Y) = \frac{1}{2}\nu[X,Y] , \quad X,Y \in \Gamma(H \oplus \{\xi\}) .$$

On the other hand, for any $U, V \in \Gamma(\mathrm{H}^{\perp})$ we have

$$\nabla_U V = \widehat{\nabla}_U V + L(U,V) \tag{13}$$

where $\widehat{\nabla}_U V = \mathcal{V}(\nabla_U V)$ and $L(U,V) = \mathcal{H}(\nabla_V X)$.
Furthermore, for vertical vector V and horizantal vector X, we get

$$\nabla_V X = \mathcal{H}(\nabla_V X) + \mathcal{T}_V X.$$

Let X be basic vector. Then, it is $[V,X] \in \Gamma(H^{\perp})$ for $V \in \Gamma(H^{\perp})$.
Hence we have

$$\mathcal{H}(\nabla_V X) = \mathcal{H}(\nabla_X V) = \mathcal{A}_X V.$$

Moreover for $V \in \Gamma(H^{\perp})$ and basic vector X, we get

$$\nabla_V X = \mathcal{A}_X V + \mathcal{T}_V X$$

Finally, the operator $\mathcal{T}$ and L related by

$$g(\mathcal{T}_V X, W) = -g(L(V,W),X) \tag{14}$$

The curvature tensors R, R^* of the connection ∇, ∇^* on $M\ and\ B$ respectively related by

$$\begin{aligned} R(X,Y,Z,W) & \\ & = R^*(X_*,Y_*,Z_*,W_*) \\ & - g\big(C(Y,Z),C(X,W)\big) \\ & + g(C(X,Z),C(Y,W)) + \\ 2g(C(X,Y),C(Z,W)) & \end{aligned} \tag{15}$$

where $X,Y,Z,W \in \Gamma(H \oplus \{\xi\})$, $\pi_* X = X_*$, $\pi_* Y = Y_*$, $\pi_* Z = Z_*$ and $\pi_* W = W_* \in \Gamma(TB)$

3. Submersions of Generic Submanifolds of a Para Kenmotsu Manifold

Definition 3.1. Let M be a generic submanifold of a para Kenmotsu manifold $\bar{M}$ and B be an almost contact metric manifold with the almost contact metric structure $(\varphi', \xi', \eta', g')$. Assume that there is a submersion $\pi : M \longrightarrow B$ such that

(i) $H^{\perp} = ker\pi_*$, where $\pi_* : TM \longrightarrow TB$ is the tangent mapping to π,

(ii) $\pi_*: H_p \oplus \{\xi\} \longrightarrow T_{\pi(p)}B$ is an isometry for each $p \in M$ which satisfies $\pi_* \circ \varphi = \varphi' \circ \pi_*$; $\eta = \eta' \circ \pi_*$; $\pi_*(\xi_p) = \xi'_{\pi(p)}$, where $T_{\pi(p)}B$ denotes the tangent space of B at $\pi(p)$.

Proposition 3.2. Let $\pi : (M, g_M) \to (B, g_B)$ be a submersion of generic submanifold of a para Kenmotsu manifold $\bar{M}$ onto an almost contact metric manifold B. Then we have

$$C(X,\varphi Y) = PC(X,Y) + th(X,Y),$$
$$h(X,\varphi Y) = FC(X,Y) + fh(X,Y)$$

for any $X, Y \in \Gamma(H)$.
Proof. For all $X, Y \in \Gamma(H)$ using (3) and (4) we have

$$\nabla_X \varphi Y + h(X,\varphi Y) = \varphi \nabla_X Y + \varphi h(X,Y) +$$

$g(\varphi X, Y)\xi - \eta(Y)\varphi X.$
Then, using (12) we have

$$\begin{aligned}\widetilde{\nabla}^{*}_{X}\varphi Y + C(X,\varphi Y) + h(X,\varphi Y) &\\ &= \varphi\widetilde{\nabla}^{*}_{X}Y + PC(X,Y) + FC(X,Y)\\ &+ g(PX,Y)\xi + g(FX,Y)\xi\\ &- \eta(Y)PX - \eta(Y)FX\end{aligned}$$

Comparing components on both sides in the above equation, we get the required results.

Corollary 3.3. Let $\pi: M \longrightarrow B$ be a submersion of generic submanifold of a para Kenmotsu manifold $\overline{M}$ onto an almost contact metric manifold.Then we have

$$C(X,\varphi Y) + h(X,\varphi Y) = \varphi C(X,Y) + \varphi h(X,Y)$$

for any $X, Y \in \Gamma(H)$.

Theorem 3.4. Let $\pi: (M, g_M) \to (B, g_B)$ be a submersion of generic submanifold of a para Kenmotsu manifold $\overline{M}$ onto an almost contact metric manifold B. Then we have

$$L(V,PW) - \mathcal{H}(\tilde{\mathcal{A}}_{FW}V) = \varphi L(V,W) \tag{16}$$

$$\begin{aligned}\widehat{\nabla}_V PW - \mathcal{V}(\tilde{\mathcal{A}}_{FW}V) &\\ &= P\widehat{\nabla}_V W + h(V,W) + g(PV,W)\xi\\ &- \eta(W)PV\end{aligned}$$

$$\begin{aligned}h(V,PW) + \nabla^{\perp}_{V}FW &\\ &= F\widehat{\nabla}_V W + fh(V,W) - \eta(W)FW\end{aligned}$$

for all $V, W \in \Gamma(\mathrm{H}^{\perp})$.
Proof. For all $V, W \in \Gamma(\mathrm{H}^{\perp})$, using (3) and (8) we have

$$\overline{\nabla}_V PW + \overline{\nabla}_V FW = \varphi\overline{\nabla}_V W + g(\varphi V, W)\xi - \eta(W)\varphi V.$$

Moreover, for (4) and (5) we get

$$\begin{aligned}\nabla_V PW + h(V,PW) + (-\tilde{\mathcal{A}}_{FW}V) + \nabla^{\perp}_{V}FW &\\ &= \varphi(\nabla_V W + h(V,W))\\ &+ g(PV,W)\xi + g(FV,W)\xi\\ &- \eta(W)PV - \eta(W)FV\end{aligned}$$

Then using (13) we have

$$\begin{aligned}\widehat{\nabla}_V PW + L(V,PW) + h(V,PW) - \mathcal{H}(\tilde{\mathcal{A}}_{FW}V) &\\ &+ \nabla^{\perp}_{V}FW - \mathcal{V}(\tilde{\mathcal{A}}_{FW}V)\\ &= \varphi(\widehat{\nabla}_V W + L(V,W) + h(V,W)\\ &+ g(PV,W)\xi - \eta(W)PV\end{aligned}$$

Finally, for (8) we obtain

$$\begin{aligned}\widehat{\nabla}_V PW + L(V,PW) + h(V,PW) - \mathcal{H}(\tilde{\mathcal{A}}_{FW}V) &\\ &+ \nabla^{\perp}_{V}FW - \mathcal{V}(\tilde{\mathcal{A}}_{FW}V)\\ &= P\widehat{\nabla}_V W + h(V,W) + \varphi\widehat{\nabla}_V W\\ &+ \varphi L(V,W) + \varphi h(V,W)\\ &+ \varphi g(PV,W)\xi - \varphi\eta(W)PV\\ &+ F\widehat{\nabla}_V W + fh(V,W) - \eta(W)FW\end{aligned}$$

Comparing components of horizantal and vertical on both sides in the above equation, we get the required results.

Theorem 3.5. Let $\pi: (M, g_M) \to (B, g_B)$ be a submersion of generic submanifold of a para Kenmotsu manifold $\overline{M}$ onto an almost contact metric manifold B. Then we have

$$\mathcal{A}_{\varphi X}V = \varphi\mathcal{A}_X V$$

for all $X \in \Gamma(H)$ and $V \in \Gamma(\mathrm{H}^{\perp})$.
Proof. X is basic vector and for $Y \in \Gamma(H)$ ve $V \in \Gamma(\mathrm{H}^{\perp})$ we get

$$g(\mathcal{A}_{\varphi X}V, Y) = g(\nabla_{\varphi X}V, Y) - g(\mathcal{V}\nabla_{\varphi X}V, Y)$$

Moreover, using $g(\mathcal{V}\nabla_{\varphi X}V, Y) = 0$ and ∇ lineer connection we have

$$g(\mathcal{A}_{\varphi X}V, Y) = g(\nabla_V\varphi X, Y).$$

Then for (4) we obtain

$$g(\mathcal{A}_{\varphi X}V, Y) = g(\varphi\overline{\nabla}_V X, Y).$$

On the other hand using (2) and (5) we have

$$g(\mathcal{A}_{\varphi X}V, Y) = -g(\mathcal{A}_X V, \varphi Y)$$

which complate proof.

Proposition 3.6. Let $\pi: (M, g_M) \to (B, g_B)$ be a submersion of generic submanifold of a para Kenmotsu manifold $\overline{M}$ onto an almost contact metric manifold B. Then we have

$$C(\varphi X, \varphi Y) = C(X,Y)$$

and

$$C(X, \varphi Y) = -C(\varphi X, Y)$$

for all $X, Y \in \Gamma(H)$.

Theorem 3.7. Let $\pi: (M, g_M) \to (B, g_B)$ be a submersion of generic submanifold of a para Kenmotsu manifold $\bar{M}$ onto an almost contact metric manifold B. Then B is also a para Kenmotsu manifold.

Proof. For any $X, Y \in \Gamma(TM)$, using (15), we get

$$(\tilde{\nabla}_X^* \varphi)Y = g_M(\varphi X, Y)\xi - \eta(Y)\varphi X$$

Applying π_* to the above equation and using Lemma 2.1., (11) and definition of submersion, we derive

$$\left(\tilde{\nabla}_{X_*}^* \varphi'\right)Y_* = g_B(\varphi' X_*, Y_*)\xi' - \eta'(Y_*)\varphi' X_*$$

The above equation shows that B is a para Kenmotsu manifold.

4. Curvature Relations

Firstly we determine $\bar{P}$ and P^* holomorphic sectional curvature, para Kenmotsu manifold $\bar{M}$ and almost contact metric manifold B, respectively which $S(X,Y) = R(X, \varphi X, Y, \varphi Y)$.

Using (7) and (13) we have

$$\begin{gathered}\bar{R}(X, \varphi X, Z, \varphi Z) = R^*(X_*, \varphi X_*, Z_*, \varphi Z_*) + \\ g\big(C(X,Z), C(\varphi X, \varphi Z)\big) - g\big(C(\varphi X, Z), C(X, \varphi Z)\big) + \\ 2g(C(X, \varphi X), C(Z, \varphi Z)) - \\ g\big(h(X, \varphi Z), h(\varphi X, Z)\big) + g(h(X,Z), h(\varphi X, \varphi Z)\end{gathered} \tag{17}$$

Theorem 4.1. Let $\pi: (M, g_M) \to (B, g_B)$ be a submersion of generic submanifold of a para Kenmotsu manifold $\bar{M}$ onto an almost contact metric manifold B. Then the bisectional curvature $\bar{K}$ and K^* of $\bar{M}$ and B,respectively satisfy

$$\bar{K}(W,X) = K^*(W_*, X_*) + \|h(X, \varphi W)\|^2 + \|h(X,W)\|^2$$

for all $\forall\, X, W \in \Gamma(H)$.

Proof. For all $X, W \in \Gamma(H)$, we have

$$h(X, \varphi W) = h(\varphi X, W) = \varphi h(X,W)$$

Using Corollary 3.3. we get

$$2h(X, \varphi W) = 2\varphi C(X,W) + 2\varphi h(X,W)$$

or

$$C(X,W) = 0$$

Then, for (7) we have

$$\begin{aligned}&\bar{R}(X, \varphi X, W, \varphi W) \\ &= R^*(X_*, \varphi X_*, W_*, \varphi W_*) \\ &- g\big(h(X, \varphi W), h(X, \varphi W)\big) \\ &+ g(h(X,W), -h(X,W)\end{aligned}$$

which complate proof.

Corollary.4.2.2. Let $\pi: (M, g_M) \to (B, g_B)$ be a submersion of generic submanifold of a para Kenmotsu manifold $\bar{M}$ onto an almost contact metric manifold B. Then holomorphic sectional curvature $\bar{P}$ *ve* $P^*$of $\bar{M}$ *ve* B respectively satisfy

$$\bar{S}(X) = S^*(X_*) + \|h(X, \varphi X)\|^2 + \|h(X,X)\|^2$$

for all $X \in \Gamma(H)$.

5. Totally Geodesic Fibres

Definition 5.1. Let M be generic submanifold of para Kenmotsu manifold $\bar{M}$. Then M is called mixed totally geodesic if $h(X,V) = 0$, for all $X \in \Gamma(H)$ *and* $V \in \Gamma(H^\perp)$

Theorem 5.2. Let $\pi: (M, g_M) \to (B, g_B)$ be a submersion of generic submanifold of a para Kenmotsu manifold $\bar{M}$ onto an almost contact metric manifold B. Then M is mixed totally geodesic if and only if

$$A_N V \in \Gamma(H^\perp)$$

for all $V \in \Gamma(H^\perp)$ *ve* $N \in \Gamma(T^\perp M)$.

Proof. Suppose that M is mixed totally geodesic. For all $X, Y \in \Gamma(\mathrm{TM})$*ve* $N \in \Gamma(TM^\perp)$, using (4)-(6) we have $g(A_N V, X) = 0$. Then we conclude $A_N V \in \Gamma(H^\perp)$.

On the other hand, for $A_N V \in \Gamma(H^\perp)$and $X \in \Gamma(H)$ using (6) we get $h(V,X) = 0$ which complate proof.

Theorem 5.3. Let $\pi: (M, g_M) \to (B, g_B)$ be a submersion of generic submanifold of a para Kenmotsu manifold $\bar{M}$ onto an almost contact metric manifold B and $v = 0$. Then M is mixed totally geodesic if fibers is totally geodesic submanifold of M.

Proof. Let fibers are totally geodesic. Then for all $V \in \Gamma(H^\perp), X \in \Gamma(\mathrm{TM})$ we have $\mathcal{T}_V X = 0$. Then using (14) we get $L(V,W) = 0$.

Moreover for (16) we obtain $\mathcal{H}(\tilde{\mathcal{A}}_N V) = 0$ or $\tilde{\mathcal{A}}_N V \in \Gamma(H^\perp)$.

Finally using theorem 5.2.. which complate proof.

Theorem 5.4. Let $\pi: (M, g_M) \to (B, g_B)$ be a submersion of generic submanifold of a para Kenmotsu manifold $\bar{M}$ onto an almost contact metric manifold B with totally geodesic fibers. Then we have

$\bar{R}(X, V, Y, W) = -g\big((\nabla_V C)(X,Y), W\big) +$

$g(\mathcal{A}_X V, \mathcal{A}_Y W) - g(h(X,Y), h(V,W))$
for all $X, Y \in \Gamma(H)$ and $V, W \in \Gamma(H^{\perp})$.

Proof. For all $X, Y \in \Gamma(H)$ and $V, W \in \Gamma(H^{\perp})$, using (10) we get

$$R(V,X,Y,W) = g\big((\nabla_X L)(V,W),Y\big) + g\big((\nabla_V C)(X,Y),W\big) + g(\mathcal{A}_X V,\mathcal{A}_Y W) - g(\mathcal{T}_V X,\mathcal{T}_W Y) \qquad (18)$$

Then for (7) and (18) we have

$$\begin{aligned}\bar{R}(X,V,Y,W) = &-g\big((\nabla_X L)(V,W),Y\big)\\ &+ g\big((\nabla_V C)(X,Y),W\big)\\ &- g(\mathcal{A}_X V,\mathcal{A}_Y W) - g(\mathcal{T}_V X,\mathcal{T}_W Y)\\ &- g\big(h(X,W),h(V,Y)\big)\\ &+ g(h(X,Y),h(V,W))\end{aligned}$$

On the other hand, since fibers are totally geodesic and theorem 5.3, we obtain

$$g(\mathcal{T}_V X,\mathcal{T}_W Y) = -g\big(X, L(V,\mathcal{V}\nabla_W Y)\big) = 0 \text{ and}$$
$$h(X,V) = 0$$

which complete proof.

Corollary 5.5. Let $\pi: (M, g_M) \to (B, g_B)$ be a submersion of generic submanifold of a para Kenmotsu manifold $\bar{M}$ onto an almost contact metric manifold B. Let fibers are totally geodesic submanifolds of M. Then we have

$$\bar{K}(X \wedge V) = g\big(h(X,X),h(V,V)\big) - \|\mathcal{A}_X V\|^2$$

for all $X, Y \in \Gamma(H)$ and $V, W \in \Gamma(H^{\perp})$.

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