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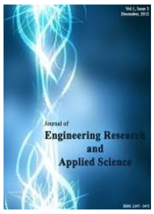

Some Characterizations Invariant Submanifolds of A (κ,μ)-Para Contact Space

Pakize Uygun[a]*, Suleyman Dirik[b], Mehmet Atceken[c], Tugba Mert[d]

[a,*] Tokat Gaziosmanpasa University, Faculty of Arts and Sciences, Mathematics, Tokat, Turkey.

[b] Amasya University, Faculty of Arts and Sciences, Mathematics, Amasya, Turkey.

[c] Aksaray University, Faculty of Arts and Sciences, Mathematics, Aksaray, Turkey.

[d] Sivas Cumhuriyet University, Faculty of Arts and Sciences, Mathematics, Sivas, Turkey.

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Abstract
The aim of this present paper is to study certain conditions for an invariant submanifold of a (κ,μ) $-$paracontact space. We classify (κ,μ) $-$paracontact space form satisfying the curvature conditions $W_8\cdot\sigma=0$, $W_7\cdot\sigma=0$ and $W_7\cdot\nabla\sigma=0$. Recently, we reach at these conditions are equivalent to $\sigma=0$.

Keywords: (κ,μ) $-$Paracontact metric manifold, Invariant submanifold, semiparallel submanifold, 2-semiparallel submanifold.

1. Introduction

The geometry of submanifolds has grown rapidly in current geometry as a result of its numerous applications in practical mathematics. However, in the theory of relativity, the concept of geodesics plays a crucial role. Furthermore, rather than the simplest submanifolds, totally geodesic submanifolds are extremely significant in physical sciences.

S. Kaneyuki and F. Williams researched the paracomplex analogs of almost contact structures and D $-$homothetic transformations as a special conformal transformation and the behavior of the Einstein condition under D $-$homothetic transformations on a paracontact metric manifold [22].

Also, M. Atçeken et al. studied invariant semiparallel and 2-semiparallel submanifolds in a normal paracontact metric manifold. They established both the required and sufficient requirements for the submanifold to be totally geodesic [4]. K. Arslan et al. introduced 2-semiparallel surfaces satisfying the integrability condition of the differential system. They did classification of non-totally geodesic and

the notion of parahodge structures on manifolds. They constructed new examples of paracomplex manifolds and all simply connected parahermitian symmetric coset spaces [9]. S. Zamkovoy studied the properties of an almost paracontact metric manifold. They searched conformal transformations of a paracontact manifold i.e. transformations preserving the paracontact structure. In addition, they defined

non-totally umbilical 2-semiparallel surfaces are given a space form for two particular cases: surfaces with a flat normal connection $\nabla^{\perp}$ and for pointwise isotropic surfaces [1]. Many geometers, inspired by these studies, studied invariant submanifolds of various manifolds. [3],[2].

Motivated by the above studies, we search for an invariant submanifold of a (κ,μ)-paracontact space. In this paper, we investigate the conditions $W_8\cdot\sigma=0$, $W_7\cdot\sigma=0$ and $W_7\cdot\nabla\sigma=0$ for an invariant submanifold of a (κ,μ) $-$paracontact space.

Corresponding author: pakizeuygun@hotmail.com

2. Preliminaries

0.8cm A $(2n+1)$-dimensional smooth manifold $\widetilde{M}$ is said to be a paracontact metric manifold if it admits a (1,1)-type tensor field ϕ, a unit vector field ξ, 1-form η, and a semi-Riemannian metric tensor g which satisfy

$$\phi^2\alpha_1 = \alpha_1 - \eta(\alpha_1)\xi, \quad \eta(\alpha_1) = g(\alpha_1, \xi) \tag{2.1}$$

$$g(\phi\alpha_1, \phi\alpha_2) = -g(\alpha_1, \alpha_2) + \eta(\alpha_1)\eta(\alpha_2), \quad \eta \circ \phi = 0, \tag{2.2}$$

and

$$d\eta(\alpha_1, \alpha_2) = g(\alpha_1, \phi\alpha_2),$$

for all $\alpha_1, \alpha_2 \in \Gamma(T\widetilde{M})$, where $\Gamma(T\widetilde{M})$ denote the set of the differentiable vector fields on $\widetilde{M}$ [16].

In a paracontact metric manifold $(\widetilde{M}, \phi, \eta, \xi, g)$, we define a (1,1)-type tensor field by h. One can easily to see that h is a symmetric and satisfies

$$h\xi = 0, \quad h\phi = -\phi h \quad and \quad Trh = 0. \tag{2.3}$$

Moreover, for a (κ, μ) $-$paracontact metric manifold $\widetilde{M}$ of dimensional $(2n+1)$ and for all $\alpha_1, \alpha_2 \in \Gamma(T\widetilde{M})$, we have

$$(\widetilde{\nabla}_{\alpha_1}\phi)\alpha_2 = -g(\alpha_1 - h\alpha_1, \alpha_2)\xi + \eta(\alpha_2)(\alpha_1 - h\alpha_1), \tag{2.4}$$

where $\widetilde{\nabla}$ denotes the Riemannian connection with respect to g. From (2.4), taking instead of ξ

$$\widetilde{\nabla}_{\alpha_1}\xi = -\phi\alpha_1 + \phi h\alpha_1, \tag{2.5}$$

for all $\alpha_1 \in \Gamma(T\widetilde{M})$ [22].

A paracontact metric manifold $\widetilde{M}^{2n+1}(\phi, \xi, \eta, g)$ is said to be a (κ, μ)-space form if its the Riemannian curvature tensor $\tilde{R}$ satisfies

$$\tilde{R}(\alpha_1, \alpha_2)\xi = \kappa\{\eta(\alpha_2)\alpha_1 - \eta(\alpha_1)\alpha_2\} + \mu\{\eta(\alpha_2)h\alpha_1 - \eta(\alpha_1)h\alpha_2\}, \tag{2.6}$$

for all $\alpha_1, \alpha_2 \in \Gamma(T\widetilde{M})$, where κ, μ are real constant [9]. The geometric structure of the (κ, μ)-paracontact metric manifold varies with $\kappa < -1$, $\kappa = -1$, and $\kappa > -1$. In addition, for the cases $\kappa < -1$ and $\kappa > -1$, (κ, μ)-nullity condition (2.6) entirely specifies the curvature tensor field [14].

In a (κ, μ)-paracontact metric manifold $\widetilde{M}^{2n+1}(\phi, \xi, \eta, g)$, we have

$$S(\alpha_1, \alpha_2) = [2(1-n) + n\mu]g(\alpha_1, \alpha_2) + [2(n-1) + \mu]g(h\alpha_1, \alpha_2)$$
$$+[2(n-1) + n(2\kappa - \mu)]\eta(\alpha_1)\eta(\alpha_2), \tag{2.7}$$

$$S(\alpha_1, \xi) = 2nk\eta(\alpha_1), \quad Q\xi = 2nk\xi, \tag{2.8}$$

$$h^2 = (1+\kappa)\phi^2, \tag{2.9}$$

$$Q\phi - \phi Q = 2[2(n-1) + \mu]h\phi, \tag{2.10}$$

for all $\alpha_1, \alpha_2 \in \Gamma(T\widetilde{M})$, where S and Q denote the Ricci tensor and Ricci operator defined $S(\alpha_1, \alpha_2) = g(Q\alpha_1, \alpha_2)$.

On a semi-Riemannian manifold (M, g), for a (o, k)-type tensor field T and (0,2)-type tensor field A, $(0, k+2)$-type tensor field $Q(A, T)$ is defined as

$$Q(A,T)(\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1k}; \alpha_1, \alpha_2) = -T((\alpha_1 \wedge_A \alpha_2)\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1k})$$
$$-T(\alpha_{11}, (\alpha_1 \wedge_A \alpha_2)\alpha_{1,}\alpha_{13}, \ldots, \alpha_{1k})$$
$$.$$
$$.$$
$$.$$
$$-T(\alpha_{11}, \alpha_{12}, \ldots, (\alpha_1 \wedge_A \alpha_2)\alpha_{1k}), \tag{2.11}$$

for all $\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1k}, \alpha_1, \alpha_2 \in \Gamma(TM)$, where

$$(\alpha_1 \wedge_A \alpha_2)\alpha_{11} = A(\alpha_2, \alpha_{11})\alpha_1 - A(\alpha_1, \alpha_{11})\alpha_2. \tag{2.12}$$

The W_7 $-$curvature tensor and W_8 $-$curvature tensor of a Riemannian manifold $\widetilde{M}^{2n+1}(\phi, \xi, \eta, g)$ are, respectively, given by

$$W_7(\alpha_1, \alpha_2)\alpha_3 = R(\alpha_1, \alpha_2)\alpha_3 - \frac{1}{2n}[S(\alpha_2, \alpha_3)\alpha_1 - g(\alpha_2, \alpha_3)Q\alpha_1] \tag{2.13}$$

and

$$W_8(\alpha_1, \alpha_2)\alpha_3 = R(\alpha_1, \alpha_2)\alpha_3 - \frac{1}{2n}[S(\alpha_2, \alpha_3)\alpha_1 - S(\alpha_1, \alpha_2)\alpha_3] \tag{2.14}$$

for all $\alpha_1, \alpha_2, \alpha_3 \in \Gamma(T\widetilde{M})$[15].

3. Invariant Submanifolds of A (κ,μ)-Paracontact Metric Manifold

Now, let M be an immersed submanifold of a (κ,μ)-paracontact metric manifold $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$, by ∇ and $\nabla^{\perp}$, we denote the induced connections on $\Gamma(TM)$ and $\Gamma(T^{\perp}M)$, respectively. Then the Gauss and Weingarten formulas are, respectively, given by

$$\widetilde{\nabla}_{\alpha_1}\alpha_2=\nabla_{\alpha_1}\alpha_2+\sigma(\alpha_1,\alpha_2) \tag{3.1}$$

and

$$\widetilde{\nabla}_{\alpha_1}\alpha_5=-A_{\alpha_5}\alpha_1+\nabla^{\perp}_{\alpha_1}\alpha_5, \tag{3.2}$$

for all $\alpha_1,\alpha_2\in\Gamma(TM)$ and $\alpha_5\in\Gamma(T^{\perp}M)$, where σ and A are called the second fundamental form and shape operator of M, respectively[3]. They are related by

$$g(\sigma(\alpha_1,\alpha_2),\alpha_5)=g(A_{\alpha_5}\alpha_1,\alpha_2).$$

If $\widetilde{\nabla}_{\alpha_1}\sigma=0$, then the submanifold is said to be parallel to the second fundamental form. The covariant derivatives of σ and A_{α_5} are defined by,

$$(\widetilde{\nabla}_{\alpha_1}\sigma)(\alpha_2,\alpha_3)=\nabla^{\perp}_{\alpha_1}\sigma(\alpha_2,\alpha_3)-\sigma(\nabla_{\alpha_1}\alpha_2,\alpha_3)-\sigma(\alpha_2,\nabla_{\alpha_1}\alpha_3), \tag{3.3}$$

and

$$(\widetilde{\nabla}_{\alpha_1}A)_{\alpha_5}\alpha_2=\nabla_{\alpha_1}A_{\alpha_5}\alpha_2-A_{\nabla^{\perp}_{\alpha_1}\alpha_5}-A_{\alpha_5}\nabla_{\alpha_1}\alpha_2. \tag{3.4}$$

For an immersed submanifold M of a (κ,μ)-paracontact metric manifold $\widetilde{M}^{2n+1}(\phi,\eta,\xi,g)$, M is said to be invariant if the structure vector field ξ is tangent to M at every point of M and $\phi\alpha_1$ is tangent to M for all $\alpha_1\in\Gamma(TM)$ at every point on M, that is, $\phi(T_{\alpha_1}M)\subseteq T_{\alpha_1}M$ at each point $\alpha_1\in M$. In the remainder of this work, we shall assume that M is an invariant submanifold unless otherwise stated.

Lemma 3.1 *Let M be an invariant submanifold of a (κ,μ)-paracontact metric manifold $\widetilde{M}^{2n+1}(\phi,\eta,\xi,g)$. Then the following relations hold.*

$$\nabla_{\alpha_1}\xi=-\phi\alpha_1+\phi h\alpha_1 \tag{3.5}$$

$$\sigma(\alpha_1,\xi)=0, \tag{3.6}$$

$$\sigma(\phi\alpha_1,\alpha_2)=\sigma(\alpha_1,\phi\alpha_2)=\phi\sigma(\alpha_1,\alpha_2), \tag{3.7}$$

for all $\alpha_1,\alpha_2\in\Gamma(TM)$.

Proof. Since the proof is a result of direct calculations.

Now, we will consider the curvature tensor W_7 of (κ,μ)-paracontact metric manifold form for later use. From (2.13) and (2.6), we have

$$\begin{aligned}W_7(\xi,\alpha_2)\alpha_3&=R(\xi,\alpha_2)\alpha_3-\frac{1}{2n}S(\alpha_2,\alpha_3)\xi+\kappa g(\alpha_2,\alpha_3)\xi\\&=\kappa g(\alpha_2,\alpha_3)\xi-\eta(\alpha_3)\alpha_2]+\mu g(h\alpha_2,\alpha_3)\xi-\eta(\alpha_3)h\alpha_2]\\&-\frac{1}{2n}S(\alpha_2,\alpha_3)\xi+\kappa g(\alpha_2,\alpha_3)\xi.\end{aligned} \tag{3.8}$$

If we choose $\alpha_1=\xi$ in (2.14), we get

$$\begin{aligned}W_8(\xi,\alpha_2)\alpha_3&=R(\xi,\alpha_2)\alpha_3-\frac{1}{2n}S(\alpha_2,\alpha_3)\xi+\kappa\eta(\alpha_2)\alpha_3\\&=\kappa g(\alpha_2,\alpha_3)\xi-\eta(\alpha_3)\alpha_2]+\mu g(h\alpha_2,\alpha_3)\xi-\eta(\alpha_3)h\alpha_2]\\&-\frac{1}{2n}S(\alpha_2,\alpha_3)\xi+\kappa\eta(\alpha_2)\alpha_3.\end{aligned} \tag{3.9}$$

Theorem 3.1 *Let M be an invariant submanifold of a (κ,μ)-paracontact metric space form $\widetilde{M}^{2n+1}(\phi,\eta,\xi,g)$. Then M is a W_8 semiparallel if and only if M is a totally geodesic submanifold provided*

$$\kappa=\pm\sqrt{\mu^2(1+\kappa)}.$$

Proof. Suppose that M is a W_8 semiparallel submanifold, that is,

$$W_8(\alpha_1,\alpha_2)\cdot\sigma=0, \tag{3.10}$$

means that,

$$R^{\perp}(\alpha_1,\alpha_2)\sigma(\alpha_3,\alpha_4)-\sigma(W_8(\alpha_1,\alpha_2)\alpha_3,\alpha_4)-\sigma(\alpha_3,W_8(\alpha_1,\alpha_2)\alpha_4)=0, \tag{3.11}$$

for all $\alpha_1,\alpha_2,\alpha_3,\alpha_4\in\Gamma(TM)$. For $\alpha_4=\xi$ in (3.11), we arrive

$$\sigma(\alpha_3,W_8(\alpha_1,\alpha_2)\xi)=0. \tag{3.12}$$

Also, by using (3.9) in (3.12), we obtain

$$\sigma(\alpha_3,-\kappa\eta(\alpha_1)\alpha_2+\frac{1}{2n}S(\alpha_1,\alpha_2)\xi+\mu\eta(\alpha_2)h\alpha_1)-\mu\eta(\alpha_1)h\alpha_2)=0. \tag{3.13}$$

This reduces for $\alpha_1=\xi$,

$$\kappa\sigma(\alpha_3,\alpha_2)+\mu\sigma(\alpha_3,hY)=0 \tag{3.14}$$

If $h\alpha_2$ is written instead of α_2 at (3.14) and using (2.9), we obtain

$$\kappa\sigma(\alpha_3,h\alpha_2)+\mu\sigma(\alpha_3,h^2\alpha_2)=\kappa\sigma(\alpha_3,h\alpha_2)+\mu(1+\kappa)\sigma(\alpha_3,\alpha_2)=0. \tag{3.15}$$

From (3.14) and (3.15), we conclude that

$$(\mu^2(1+\kappa)-\kappa^2)\sigma(\alpha_3,\alpha_2)=0.$$

This proves our assertion.

Theorem 3.2 *Let M be an invariant submanifold of a (κ,μ)-paracontact metric space form $\widetilde{M}^{2n+1}(\phi,\eta,\xi,g)$. Then M is a W_7 semiparallel if and only if M is a totally geodesic submanifold provided* $\kappa^2-\mu^2(1+\kappa)\neq 0$.

Proof. Assume that M is a W_7 semiparallel submanifold. It follows that,

$$W_7(\alpha_1,\alpha_2)\cdot\sigma=0, \tag{3.16}$$

means that,

$$R^{\perp}(\alpha_1,\alpha_2)\sigma(\alpha_3,\alpha_4)-\sigma(W_7(\alpha_1,\alpha_2)\alpha_3,\alpha_4)-\sigma(\alpha_3,W_7(\alpha_1,\alpha_2)\alpha_4)=0, \tag{3.17}$$

for all $\alpha_1,\alpha_2,\alpha_3,\alpha_4\in\Gamma(TM)$. Using $\alpha_4=\xi$ in (3.17), we get

$$\sigma(\alpha_3,W_7(\alpha_1,\alpha_2)\xi)=0. \tag{3.18}$$

Then from (3.8) to (3.18), we obtain

$$\sigma(\alpha_3,-\kappa\eta(\alpha_1)\alpha_2+\frac{1}{2n}\eta(\alpha_2)Q\alpha_1+\mu\eta(\alpha_2)h\alpha_1-\mu\eta(\alpha_1)h\alpha_2)=0. \tag{3.19}$$

This reduces for $\alpha_1=\xi$, we reach at

$$\kappa\sigma(\alpha_3,\alpha_2)-\mu\sigma(\alpha_3,h\alpha_2)=0. \tag{3.20}$$

Substituting α_2 by $h\alpha_2$ in (3.20) and using (2.9), we have

$$\kappa\sigma(\alpha_3,h\alpha_2)-\mu\sigma(\alpha_3,h^2\alpha_2)=\kappa\sigma(\alpha_3,h\alpha_2)-\mu(1+\kappa)\sigma(\alpha_3,\alpha_2)=0. \tag{3.21}$$

From (3.20) and (3.21), we conclude that

$$(\kappa^2-\mu^2(1+\kappa))\sigma(\alpha_3,\alpha_2)=0,$$

which proves our assertion.

Theorem 3.3 *Let M be an invariant submanifold of a (κ,μ)-paracontact metric space form $\widetilde{M}^{2n+1}(\phi,\eta,\xi,g)$. Then M is a W_7 2-semiparallel if and only if M is a totally geodesic submanifold provided* $\kappa\neq 0$.

Proof. Let us suppose that M is a W_7 2-semiparallel submanifold. Then, we have

$$(W_7(\alpha_1,\alpha_2)\nabla\sigma)(\alpha_3,\alpha_4,\alpha_5)=R^{\perp}(\alpha_1,\alpha_2)(\nabla_{\alpha_3}\sigma)(\alpha_4,\alpha_5)-(\nabla_{W_7(\alpha_1,\alpha_2)\alpha_3}\sigma)(\alpha_4,\alpha_5)$$
$$-(\nabla_{\alpha_3}\sigma)(W_7(\alpha_1,\alpha_2)\alpha_4,\alpha_5)-(\nabla_{\alpha_3}\sigma)(\alpha_4,W_7(\alpha_1,\alpha_2)\alpha_5)=0, \tag{3.22}$$

for all $\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5\in\Gamma(TM)$. This statement is also true for $\alpha_1=\alpha_4=\xi$, that is,

$$R^{\perp}(\xi,\alpha_2)(\nabla_{\alpha_3}\sigma)(\xi,\alpha_5)-(\nabla_{W_7(\xi,\alpha_2)\alpha_3}\sigma)(\xi,\alpha_5)-(\nabla_{\alpha_3}\sigma)(W_7(\xi,\alpha_2)\xi,\alpha_5)$$
$$-(\nabla_{\alpha_3}\sigma)(\xi,W_7(\xi,\alpha_2)\alpha_5)=0. \tag{3.23}$$

Now, let's calculate each of these expressions.

$$R^{\perp}(\xi,\alpha_2)\{\nabla^{\perp}_{\alpha_3}\sigma(\xi,\alpha_5)-\sigma(\nabla_{\alpha_3}\xi,\alpha_5)-\sigma(\xi,\nabla_{\alpha_3}\alpha_5)\}$$

$$= R^{\perp}(\xi,\alpha_2)\{-\sigma(\nabla_{\alpha_3}\xi,\alpha_5)\}$$
$$= R^{\perp}(\xi,\alpha_2)\{\sigma(\phi\alpha_3,\alpha_5)-\sigma(\phi h\alpha_3,\alpha_5)\}. \quad (3.24)$$

$$(\nabla_{W_7(\xi,\alpha_2)\alpha_3}\sigma)(\xi,\alpha_5) = \nabla^{\perp}_{W_7(\xi,\alpha_2)\alpha_3}\sigma(\xi,\alpha_5) - \sigma(\nabla_{W_7(\xi,\alpha_2)\alpha_3}\xi,\alpha_5)$$
$$-\sigma(\nabla_{W_7(\xi,\alpha_2)\alpha_3}\alpha_5,\xi)$$
$$= -\sigma(-\phi W_7(\xi,\alpha_2)\alpha_3 + \phi h W_7(\xi,\alpha_2)\alpha_3,\alpha_5). \quad (3.25)$$

Also taking into account that (3.25), we obtain

$$(\nabla_{W_7(\xi,\alpha_2)\alpha_3}\sigma)(\xi,\alpha_5) = \sigma(\phi W_7(\xi,\alpha_2)\alpha_3 - \phi h W_7(\xi,\alpha_2)\alpha_3,\alpha_5)$$
$$= \kappa\eta(\alpha_3)\{\sigma(\phi h\alpha_2,\alpha_5)-\sigma(\phi\alpha_2,\alpha_5)\}$$
$$+\mu\eta(\alpha_3)\{(1+\kappa)\sigma(\phi\alpha_2,\alpha_5)-\sigma(\phi h\alpha_2,\alpha_5)\}. \quad (3.26)$$

$$(\nabla_{\alpha_3}\sigma)(W_7(\xi,\alpha_2)\xi,\alpha_5) = (\nabla_{\alpha_3}\sigma)(\kappa(\eta(\alpha_2)\xi-\alpha_2)-\mu h\alpha_2),\alpha_5)$$
$$= \kappa(\nabla_{\alpha_3}\sigma)(\eta(\alpha_2)\xi,\alpha_5) - \kappa(\nabla_{\alpha_3}\sigma)(\alpha_2,\alpha_5)$$
$$-\mu(\nabla_{\alpha_3}\sigma)(h\alpha_2,\alpha_5)$$
$$= \kappa\{\nabla^{\perp}_{\alpha_3}\sigma(\eta(\alpha_2)\xi,\alpha_5) - \sigma(\nabla_{\alpha_3}\eta(\alpha_2)\xi,\alpha_5) - \sigma(\nabla_{\alpha_3}\alpha_5,\eta(\alpha_2)\xi)\}$$
$$-\kappa(\nabla_{\alpha_3}\sigma)(\alpha_2,\alpha_5) - \mu(\nabla_{\alpha_3}\sigma)(h\alpha_2,\alpha_5)$$
$$= -\kappa\sigma(\nabla_{\alpha_3}\eta(\alpha_2)\xi,\alpha_5) - \kappa(\nabla_{\alpha_3}\sigma)(\alpha_2,\alpha_5) - \mu(\nabla_{\alpha_3}\sigma)(h\alpha_2,\alpha_5)$$
$$= \kappa\eta(\alpha_2)\{\sigma(\phi\alpha_3,\alpha_5)-\sigma(\phi h\alpha_3,\alpha_5)\}$$
$$-\kappa(\nabla_{\alpha_3}\sigma)(\alpha_2,\alpha_5) - \mu(\nabla_{\alpha_3}\sigma)(h\alpha_2,\alpha_5), \quad (3.27)$$

and finally,

$$(\nabla_{\alpha_3}\sigma)(\xi,W_7(\xi,\alpha_2)\alpha_5) = \nabla^{\perp}_{\alpha_3}\sigma(\xi,W_7(\xi,\alpha_2)\alpha_5) - \sigma(\xi,\nabla_{\alpha_3}W_7(\xi,\alpha_2)\alpha_5)$$
$$-\sigma(\nabla_{\alpha_3}\xi,W_7(\xi,\alpha_2)\alpha_5)$$
$$= \sigma(\phi\alpha_3,W_7(\xi,\alpha_2)\alpha_5) - \sigma(\phi h\alpha_3,W_7(\xi,\alpha_2)\alpha_5)$$
$$= \kappa\eta(\alpha_5)\{\sigma(\phi h\alpha_3,\alpha_2)-\sigma(\phi\alpha_3,\alpha_2)\}$$
$$+\mu\eta(\alpha_5)\{\sigma(\phi h\alpha_3,h\alpha_2)-\sigma(\phi\alpha_3,h\alpha_2)\}. \quad (3.28)$$

Thus (3.28), (3.27), (3.26) and (3.24) statements set in (3.23), we arrive

$$R^{\perp}(\xi,\alpha_2)\{\sigma(\phi\alpha_3,\alpha_5)-\sigma(\phi h\alpha_3,\alpha_5)\} - \kappa\eta(\alpha_3)\{\sigma(\phi h\alpha_2,\alpha_5)$$
$$-\sigma(\phi\alpha_2,\alpha_5)\} - \mu\eta(\alpha_3)\{(1+\kappa)\sigma(\phi\alpha_2,\alpha_5)-\sigma(\phi h\alpha_2,\alpha_5)\}$$
$$-\kappa\eta(\alpha_2)\{\sigma(\phi\alpha_3,\alpha_5)-\sigma(\phi h\alpha_3,\alpha_5)\} + \kappa(\nabla_{\alpha_3}\sigma)(\alpha_2,\alpha_5)$$
$$+\mu(\nabla_{\alpha_3}\sigma)(h\alpha_2,\alpha_5) - \kappa\eta(\alpha_5)\{\sigma(\phi h\alpha_3,\alpha_2)-\sigma(\phi\alpha_3,\alpha_2)\}$$
$$-\mu\eta(\alpha_5)\{\sigma(\phi h\alpha_3,h\alpha_2)-\sigma(\phi\alpha_3,h\alpha_2)\} = 0,$$

from which for $\alpha_5 = \xi$,

$$\kappa\{\sigma(\phi\alpha_3,\alpha_2)-\sigma(\phi h\alpha_3,\alpha_2)\} + \mu\{\sigma(\phi\alpha_3,h\alpha_2)-\sigma(\phi h\alpha_3,h\alpha_2)\} = 0. \quad (3.29)$$

If $h\alpha_2$ is written instead of α_2 in (3.29) and making use of (2.9), we obtain

$$\kappa\{\sigma(\phi\alpha_3,h\alpha_2)-\sigma(\phi h\alpha_3,h\alpha_2)\} + \mu\{\sigma(\phi\alpha_3,h^2\alpha_2)-\sigma(\phi h\alpha_3,h^2\alpha_2)\}$$
$$= \kappa\{\sigma(\phi\alpha_3,h\alpha_2)-\sigma(\phi h\alpha_3,h\alpha_2)\} + \mu(1+\kappa)\{\sigma(\phi\alpha_3,\alpha_2)-\sigma(\phi h\alpha_3,\alpha_2)\} = 0. \quad (3.30)$$

From (3.29) and (3.30), we conclude that

$$(\mu^2(1+\kappa)-\kappa^2)\{\sigma(\phi\alpha_3,\alpha_2)-\sigma(\phi h\alpha_3,\alpha_2)\} = 0. \quad (3.31)$$

Since $\mu^2(1+\kappa)-\kappa^2 \neq 0$, we get

$$\sigma(\phi\alpha_3,\alpha_2)-\sigma(\phi h\alpha_3,\alpha_2) = 0. \quad (3.32)$$

Here, if $h\alpha_3$ is taken instead of α_3 in (3.32) and by using (2.9), (2.1), we have

$$\sigma(\phi h\alpha_3,\alpha_2)-\sigma(\phi h^2\alpha_3,\alpha_2) = \sigma(\phi h\alpha_3,\alpha_2) - (1+\kappa)\sigma(\phi\alpha_3,\alpha_2) = 0. \quad (3.33)$$

(3.32) and (3.33) prove our assertion.

Conclusion 3.4 *The results from this paper are as follows. Let be an (2n+1)-dimensional (κ,μ)-paracontact manifold. In this case,*

W_8 is a semiparallel if and only if M is totally geodesic submanifold, provided

$$\kappa = \pm\sqrt{\mu^2(1+\kappa)}.$$

W_7 is a semiparallel if and only if M is totally geodesic submanifold , provided

$$\kappa^2 - \mu^2(1+\kappa) \neq 0.$$

W_7 is a 2-semiparallel if and only if M is totally geodesic submanifold , provided

$$\kappa \neq 0.$$

Other manifolds can be run under these conditions.

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